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On exact rates of growth and decay of solutions of a linear Volterra equation in linear viscoelasticity

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Abstract. The asymptotic behaviour of a scalar linear nonconvolution Volterra equation is investigated; the equation is that satisfied by the modes of a viscoelastic rod bending quasi-statically. A sufficient condition for the trivial solution to be asymptotic stable is given, as well as results on describing the exact rate of decay: in the case that the trivial solution is unstable, the exact rate of growth of solutions is specified.

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1 Introduction

In this paper we investigate the linear nonconvolution Volterra equation

$$y(t) = \int_0^t \frac{k(t-s)}{1-p(t)} y(s)\, ds + f(t), \quad t \geq 0. \tag{1}$$

This equation is satisfied by the modes of a viscoelastic rod bending quasi-statically, as is explained in Section 2.

It is shown here that if, $k(t) \geq 0$ for all $t \geq 0$, $\int_0^\infty k(t)\, dt < 1$ and

$$\limsup_{t\to\infty} p(t) < 1 - \int_0^\infty k(t)\, dt,$$

then $y(t) \to 0$ as $t \to \infty$, provided $f(t) \to 0$ as $t \to \infty$. The question then arises of how quickly the solution decays to zero. We answer this in the case that $p(t) \to \lambda \in [0, 1)$ as $t \to \infty$. If there is a characteristic root θ_λ satisfying

$$\int_0^\infty k(t)e^{-\theta_\lambda t}\,dt = 1 - \lambda,$$

conditions are supplied which imply that $\lim_{t\to\infty} y(t)e^{-\theta_\lambda t}$ exists. On the other hand, if there is no characteristic root, it is shown that $\lim_{t\to\infty} y(t)/k(t)$ exists if k belongs a class of functions introduced in [4]. The characteristic root θ_λ always exists and is positive if $\lambda > 1 - \int_0^\infty k(t)\,dt$.

In [1, 2] exact rates of decay of convolution Volterra equations were found, and it was seen in [1] that these results could be deduced economically from a general theorem concerning the convergence to a limit of solutions to a linear nonconvolution Volterra equation. Here that same theorem is employed to investigate the asymptotic behaviour of the nonconvolution equation (1).

2 Quasi-static bending of viscoelastic rods

There is a large literature on the stability of viscoelastic structures such as rods and shells. The subject is covered extensively in [9]. The hereditary nature of the constitutive equations gives rise to integral equations and more generally functional differential equations. Attention is confined here to the quasi-static bending of linear viscoelastic rods: inertia, shear and twist are ignored.

Consider a thin inhomogeneous linear viscoelastic rod bending in plane. Suppose that it has length l, that its ends are pinned at the same level, and that it is acted on by a horizontal compressive time-varying load $P(t)$ at the end $x = l$. Body forces and torques are neglected. If its motion prior to time 0 is known, and its motion is subsequently quasi-static, small vertical displacements $y(t, x)$ obey

$$B(x)\left(\frac{\partial^2 y}{\partial x^2}(t,x) - \int_0^\infty k(s)\frac{\partial^2 y}{\partial x^2}(t-s,x)\,ds\right) + P(t)y(t,x) = 0, \quad t \ge 0;$$
$$y(t,0) = y(t,l) = 0, \quad t \ge 0;$$
$$y(t,x) = \phi(t,x), \quad t \le 0.$$

The relaxation function of the rod is given by $G(t,x) = B(x)\{1 - \int_0^t k(s)\,ds\}$. Here $B(x) > 0$ is the instantaneous flexural rigidity. The kernel k satisfies $k(t) \ge 0$ and $\int_0^\infty k(s)\,ds < 1$; so that the viscoelastic material is solid.

The static elastic problem has an increasing sequence $\{P_n\}_{n\geq 1}$ of positive eigenvalues, with corresponding eigenfunctions $\{u_n\}_{n\geq 1}$, satisfying

$$\begin{gathered} B(x)u_n''(x) + P_n u_n(x) = 0, \quad 0 < x < l; \\ u_n(0) = u_n(l) = 0; \end{gathered}$$

and the normalization condition

$$\int_0^l u_m(x)u_n(x)\frac{1}{B(x)}\,dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

$P_1 > 0$ is Euler's elastic critical load. y is the superposition of a countably infinite number of modes, the nth mode being given by

$$y_n(t) = \int_0^l y(t,x)u_n(x)\frac{1}{B(x)}\,dx.$$

It is easily seen that

$$\left(1 - \frac{P(t)}{P_n}\right)y_n(t) - \int_0^t k(t-s)y_n(s)\,ds = \int_{-\infty}^0 k(t-s)\phi_n(s)\,ds, \qquad (2)$$

$$\phi_n(s) = \int_0^l \{\phi(s,x) - (l-x)\phi(s,0) - x\phi(s,l)\}u_n(x)\frac{1}{B(x)}\,dx, \quad s \leq 0.$$

The restriction $0 \leq P(t) < P_1$ for all $t \geq 0$, is imposed to exclude the phenomena of multiple solutions and solutions blowing up in finite time, as found in [14,15,19] for loads exceeding P_1. For such loads the dynamic equations of motion should be considered. No attempt is made to elaborate conditions on the kernel k and the initial history ϕ which would imply that the forcing function on the right-hand side of (2) has the regularity properties we require. However important and relevant papers are [5, 17], and [16] is a thorough work on the theory of functional differential equations with infinite delay.

In the case that P is constant, (2) is a linear convolution equation, and the asymptotic behaviour of its solutions can be found using Laplace transforms: see [6–9] for results obtained using this approach. Results on the asymptotic behaviour of solutions in the case that $P(t)$ is time-dependent have been obtained in [9, 19, 20]: similar results for rods composed of ageing viscoelastic materials are in [9, 11, 12].

Observe that (1) is in the same form as (2), and that $p(t)$ in (1) plays the role of $P(t)/P_n$. Our results determine the asymptotic properties of the individual modes, but we do not here combine them and deduce asymptotic properties of $t \mapsto y(t,\cdot)$.

3 Mathematical preliminaries

3.1 Nonconvolution linear Volterra equations

We summarize some properties of solutions of linear nonconvolution Volterra equations. Standard works which treat this topic include [13, 18]. In particular we consider the scalar equation

$$y(t) = \int_0^t a(t,s)y(s)\,ds + f(t), \quad t \geq 0, \tag{3}$$

where the kernel $a : \Delta \to \mathbb{R}$ is continuous on the triangular region

$$\Delta = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t\},$$

and $f : [0,\infty) \to \mathbb{R}$ is continuous. Existence and uniqueness can be established by examining the Neumann series associated with a.

1 Theorem. *There is a unique continuous solution $y : [0,\infty) \to \mathbb{R}$ of* (3).

The following standard result provides sufficient conditions for the solution of (3) to decay to zero.

2 Theorem. *Suppose that a obeys*

$$\sup_{t \geq S_1} \int_{S_0}^t |a(t,s)|\,ds < 1, \quad \textit{for some } 0 < S_0 \leq S_1, \tag{4}$$

$$\lim_{t\to\infty} \int_0^T |a(t,s)|\,ds = 0 \quad \textit{for every } T > 0. \tag{5}$$

If $f(t) \to 0$ as $t \to \infty$, then $y(t) \to 0$ as $t \to \infty$.

Later we shall consider Volterra equations for which condition (5) fails, and instead employ the following scalar version of [1, Theorem A.1]. This result is used by determine exact rates of growth and decay for solutions of (1).

3 Theorem. *Suppose that:*

$$b = \limsup_{S\to\infty}\left(\limsup_{t\to\infty} \int_0^S |a(t,t-u)|\,du\right) < 1, \tag{6}$$

$$c = \lim_{S\to\infty}\left(\lim_{t\to\infty} \int_0^S a(t,t-u)\,du\right) \quad \textit{exists}, \tag{7}$$

$$\limsup_{S\to\infty}\left(\limsup_{t\to\infty} \int_S^{t-S} |a(t,s)|\,ds\right) = 0, \tag{8}$$

and that there is some j in $L^1(0,\infty)$ such that

$$\lim_{t\to\infty} \int_0^T |a(t,s) - j(s)|\,ds = 0 \quad \textit{for every } T > 0. \tag{9}$$

If $\lim_{t\to\infty} f(t)$ *exists, then* $\lim_{t\to\infty} y(t)$ *exists and*

$$\lim_{t\to\infty} y(t) = (1-c)^{-1}\left\{\lim_{t\to\infty} f(t) + \int_0^\infty j(s)y(s)\,ds\right\}. \tag{10}$$

4 Remark. This theorem does not yield an explicit expression for the limit $\lim_{t\to\infty} y(t)$. However (10) may nevertheless be of value.

3.2 Renewal equations

A particular class of equations having the form (3) are convolution equations, for which $a(t,s) = \alpha(t-s)$: (3) then takes the form

$$y(t) = \int_0^t \alpha(t-s)y(s)\,ds + f(t), \quad t \geq 0. \tag{11}$$

The solution of this equation can be represented as

$$y(t) = \int_0^t \rho(t-s)f(s)\,ds + f(t),$$

in terms of ρ, the resolvent of α, which we define to be the solution of

$$\rho(t) = \int_0^t \alpha(t-s)\rho(s)\,ds + \alpha(t). \tag{12}$$

Assume that α is in $L^1(0,\infty) \cap C[0,\infty)$, with $\alpha(t) \geq 0$. The abscissa of convergence μ of the Laplace transform of α is given by

$$\mu := \inf\left\{\sigma : \int_0^\infty \alpha(t)e^{-\sigma t}\,dt < \infty\right\}. \tag{13}$$

There are two important cases to consider. The first is that α has a *characteristic root* $\theta \geq \mu$ such that

$$\int_0^\infty \alpha(t)e^{-\theta t}\,dt = 1; \tag{14}$$

the second is that

$$\int_0^\infty \alpha(t)e^{-\mu t}\,dt < 1. \tag{15}$$

In the first case, (12) can be multiplied by $e^{-\theta t}$ to obtain a *renewal equation* for $t \mapsto e^{-\theta t}\rho(t)$; indeed

$$e^{-\theta t}\rho(t) = \int_0^t \alpha(t-s)e^{-\theta(t-s)}e^{-\theta s}\rho(s)\,ds + e^{-\theta t}\alpha(t).$$

By imposing a few extra hypotheses, the renewal theorem can be used to infer that $e^{-\theta t}\rho(t)$ approaches a known constant as $t \to \infty$. For details see [10, Ch. XI] or [3, Ch. IV].

5 Theorem. *Suppose that* (14) *holds, that*

$$\int_0^\infty s e^{-\theta s}\alpha(s)\, ds < \infty, \tag{16}$$

and that $s \mapsto e^{-\theta s}\alpha(s)$ is directly Riemann integrable. Then

$$\lim_{t\to\infty} \rho(t)e^{-\theta t} = \frac{1}{\int_0^\infty s e^{-\theta s}\alpha(s)\, ds} > 0.$$

If however (15) holds, then

$$e^{-\mu t}\rho(t) = \int_0^t \alpha(t-s)e^{-\mu(t-s)}e^{-\mu s}\rho(s)\, ds + e^{-\mu t}\alpha(t).$$

is a *defective renewal equation* for $e^{-\mu t}\rho(t)$. Such equations were one of the motivations for [4], and in [1] a class $\mathcal{U}(\mu)$ of functions which satisfy the hypotheses of Theorem 3 of [4] was introduced. Roughly speaking, if α is in $\mathcal{U}(\mu)$ then $\alpha(t) = e^{\mu t}\delta(t)$ is the product of the exponential $e^{\mu t}$ and a slowly-decaying $\delta(t)$.

A counterpart of Theorem 5 for defective renewal equations is Theorem 5.2 of [2].

6 Theorem. *Suppose that* (15) *holds, and that α is in $\mathcal{U}(\mu)$. Then the resolvent ρ is in $\mathcal{U}(\mu)$, $\lim_{t\to\infty}\rho(t)/\alpha(t)$ exists and*

$$\lim_{t\to\infty}\frac{\rho(t)}{\alpha(t)} = \frac{1}{(1-\int_0^\infty \alpha(t)e^{-\mu t}\, dt)^2}. \tag{17}$$

The formal definition of $\mathcal{U}(\mu)$ is stated.

7 Definition. Let $\mu \in \mathbb{R}$. A function $\alpha : [0,\infty) \to \mathbb{R}$ is in $\mathcal{U}(\mu)$ if it is continuous with $\alpha(t) > 0$ for all $t \geq 0$, and

$$\int_0^\infty \alpha(t)e^{-\mu t}\, dt < \infty, \tag{18}$$

$$\lim_{t\to\infty}\int_0^t \frac{\alpha(t-s)\alpha(s)}{\alpha(t)}\, ds = 2\int_0^\infty \alpha(t)e^{-\mu t}\, dt, \tag{19}$$

$$\lim_{t\to\infty}\frac{\alpha(t-s)}{\alpha(t)} = e^{-\mu s} \quad \text{uniformly for } 0 \leq s \leq S, \text{ for all } S > 0. \tag{20}$$

If α is in $\mathcal{U}(0)$ it is termed a *subexponential* function. The nomenclature is suggested by the fact that (20) with $\mu = 0$ implies that $\alpha(t)e^{\epsilon t} \to \infty$ as $t \to \infty$, for every $\epsilon > 0$. α is *regularly varying at infinity* if $\alpha(\nu t)/\alpha(t)$ tends to a limit as $t \to \infty$ for all $\nu > 0$. It is noted in [2] that the class of subexponential functions includes all positive, continuous, integrable functions which are regularly varying at infinity. The properties $\mathcal{U}(0)$ have been extensively studied in [2, 4] and elsewhere.

If α is in $\mathcal{U}(\mu)$, then $\alpha(t) = e^{\mu t}\delta(t)$ where δ is a function in $\mathcal{U}(0)$. Simple examples of functions in $\mathcal{U}(\mu)$ are $\alpha(t) = e^{\mu t}(1+t)^{-\beta}$ for $\beta > 1$, $\alpha(t) = e^{\mu t}e^{-(1+t)^{\beta}}$ for $0 < \beta < 1$ and $\alpha(t) = e^{\mu t}e^{-t/\log(t+2)}$. The class $\mathcal{U}(\mu)$ therefore includes a wide variety of functions exhibiting exponential and slower than exponential decay: nor is the slower than exponential decay limited to a class of polynomially decaying functions.

8 Remark. It appears restrictive to specify the value of the limit in (19). But if $\alpha : [0, \infty) \to \mathbb{R}$ is a continuous function with $\alpha(t) > 0$ for all $t \geq 0$, satisfying (18) and (20), and

$$\lim_{t\to\infty} \int_0^t \frac{\alpha(t-s)\alpha(s)}{\alpha(t)}\, dt \quad \text{exists,}$$

it is shown in [4] that this limit is given by (19).

Proposition 3 of [1] is used later, and helps to make the proof of Theorem 14 succinct.

9 Lemma. *Let μ be in $\mathbb{R}$. Suppose that $\alpha : [0, \infty) \to \mathbb{R}$ is a continuous function with $\alpha(t) > 0$ for all $t \geq 0$, satisfying* (18) *and* (20)*. Then α is in $\mathcal{U}(\mu)$ if and only if*

$$\lim_{S\to\infty} \left(\lim_{t\to\infty} \int_S^{t-S} \frac{\alpha(t-s)\alpha(s)}{\alpha(t)}\, ds \right) = 0. \tag{21}$$

4 Stability result

In this section, the asymptotic behaviour of the solutions of (1) is investigated under the following hypotheses, which are assumed to hold hereinafter.

(H1) $p : [0, \infty) \to \mathbb{R}$ is continuous with $0 \leq p(t) < 1$ for all $t \geq 0$ and $\limsup_{t\to\infty} p(t) < 1$;

(H2) the kernel $k : [0, \infty) \to (0, \infty)$ is continuous, integrable, and

$$\int_0^\infty k(s)\, ds < 1; \tag{22}$$

(H3) $f : [0,\infty) \to \mathbb{R}$ is continuous, and $f(t) \to 0$ as $t \to \infty$.

Notice that (1) has the same form as (3) if

$$a(t,s) = \frac{k(t-s)}{1-p(t)}, \quad (t,s) \in \Delta. \tag{23}$$

Gurtin and Reynolds (cf. Theorem A2.1 of [20]) established the following result. Here we show that it is a corollary of Theorem 2.

10 Theorem. *Suppose that*

$$\limsup_{t\to\infty} p(t) < 1 - \int_0^\infty k(s)\,ds. \tag{24}$$

Then $y(t) \to 0$ *as* $t \to \infty$.

Proof. We verify that (4) and (5) hold. Firstly note that (24) is equivalent to

$$\int_0^\infty k(s)\,ds < 1 - \limsup_{t\to\infty} p(t) = \liminf_{t\to\infty}(1-p(t)),$$

and hence

$$1 > \frac{\int_0^\infty k(s)\,ds}{\liminf_{t\to\infty}(1-p(t))} = \limsup_{t\to\infty} \frac{\int_0^t k(t)\,dt}{1-p(t)}.$$

Therefore there is $S_1 > 0$ such that

$$\frac{1}{1-p(t)} \int_0^t k(s)\,ds < 1 \quad \text{for all } t \ge S_1.$$

By the definition (23), for $t \ge S_1$,

$$\begin{aligned}\int_0^t |a(t,s)|\,ds &= \frac{1}{1-p(t)} \int_0^t k(t-s)\,ds \\ &= \frac{1}{1-p(t)} \int_0^t k(\sigma)\,d\sigma < 1.\end{aligned}$$

Thus (4) is true with $S_0 = 0$. Let $T > 0$. Then, for $t \ge T$,

$$\int_0^T |a(t,s)|\,ds = \frac{1}{1-p(t)} \int_0^T k(t-s)\,ds = \frac{1}{1-p(t)} \int_{t-T}^t k(\sigma)\,d\sigma.$$

By taking the limit superior as $t \to \infty$ of each side, we deduce that

$$\limsup_{t\to\infty} \int_0^T |a(t,s)|\,ds \le \limsup_{t\to\infty} \frac{1}{1-p(t)} \lim_{t\to\infty} \int_{t-T}^t k(\sigma)\,d\sigma = 0,$$

since k is in $L^1(0,\infty)$. Hence (5) is also true. $\boxed{QED}$

11 Remark. This result can also be proved by applying Theorem 3.

5 Rates of growth and decay

In this section we assume that

$$p(t) \to \lambda \quad \text{as } t \to \infty, \tag{25}$$

and find the exact rates of *decay* as $t \to \infty$ of solutions if

$$0 \le \lambda < 1 - \int_0^\infty k(t)\,dt, \tag{26}$$

and the exact rates of *growth* as $t \to \infty$ if

$$1 - \int_0^\infty k(t) \le \lambda < 1. \tag{27}$$

Let μ be the abscissa of convergence of the Laplace transform of k, defined in (13). Due to (22), $\mu \le 0$. We investigate two cases: firstly there is a $\theta_\lambda \ge \mu$ such that

$$\int_0^\infty k(t)e^{-\theta_\lambda t}\,dt = 1 - \lambda; \tag{28}$$

secondly

$$\int_0^\infty k(t)e^{-\mu t}\,dt < 1 - \lambda. \tag{29}$$

If (28) holds, we observe that

$$\theta_\lambda = \begin{cases} < 0, & \lambda < 1 - \int_0^\infty k(s)\,ds, \\ > 0, & \lambda > 1 - \int_0^\infty k(s)\,ds. \end{cases}$$

Of course θ_λ is the characteristic root of $k/(1-\lambda)$.

At this point we derive a nonconvolution equation to which we can apply Theorem 3 if (28) is true. The case when (29) holds is deferred to later in this section. We rearrange (1) as

$$(1-\lambda)y(t) = \int_0^t k(t-s)y(s)\,ds + (p(t)-\lambda)y(t) + (1-p(t))f(t). \tag{30}$$

The resolvent r_λ of $k/(1-\lambda)$ is the solution of

$$r_\lambda(t) = \int_0^t \frac{k(t-s)}{1-\lambda} r_\lambda(s)\,ds + \frac{k(t)}{1-\lambda}. \tag{31}$$

We take the convolution of each term in (30) with r_λ, employ Fubini's theorem and simplify the result using (31), to obtain the new nonconvolution Volterra equation

$$y(t) = \int_0^t \frac{p(s)-\lambda}{1-p(t)} r_\lambda(t-s)y(s)\,ds + \int_0^t \frac{1-p(s)}{1-p(t)} r_\lambda(t-s)f(s)\,ds + f(t). \quad (32)$$

It might be expected that, since p satisfies (25), the solution y of (1) decays or grows at the same rate as that of

$$(1-\lambda)z(t) = \int_0^t k(t-s)z(s)\,ds + (1-p(t))f(t). \quad (33)$$

By taking the convolution of this equation with r_λ, and simplifying using (31), we get the representation

$$z(t) = \int_0^t r_\lambda(t-s)\frac{1-p(s)}{1-\lambda} f(s)\,ds + \frac{1-p(t)}{1-\lambda} f(t). \quad (34)$$

Hence (32) becomes

$$y(t) = \int_0^t \frac{p(s)-\lambda}{1-p(t)} r_\lambda(t-s)y(s)\,ds + \frac{1-\lambda}{1-p(t)} z(t). \quad (35)$$

Our first result gives conditions for $y(t)$ to grow or decay at the same rate as $e^{\theta_\lambda t}$ as $t \to \infty$.

12 Theorem. *Let $0 \le \lambda < 1$. Suppose that (25) holds, with*

$$\int_0^\infty |p(s)-\lambda|\,ds < \infty. \quad (36)$$

Assume that there is a $\theta_\lambda \in \mathbb{R}$ satisfying (28), that

$$\int_0^\infty s e^{-\theta_\lambda s} k(s)\,ds < \infty, \quad (37)$$

and $s \mapsto e^{-\theta_\lambda s}k(s)$ is directly Riemann integrable. If

$$\int_0^\infty e^{-\theta_\lambda t}|f(t)|\,dt < \infty, \quad (38)$$

and $e^{-\theta_\lambda t}f(t) \to 0$ as $t \to \infty$, then $\lim_{t\to\infty} e^{-\theta_\lambda t}y(t)$ exists.

Proof. We begin by noting that a consequence of Theorem 5 is

$$\lim_{t\to\infty} r_\lambda(t)e^{-\theta_\lambda t} = L_\lambda := \frac{1-\lambda}{\int_0^\infty se^{-\theta s}k(s)\,ds} > 0. \tag{39}$$

By multiplying (35) by $e^{-\theta_\lambda t}$, we get the equation

$$\tilde{y}(t) = \int_0^t \tilde{a}(t,s)\tilde{y}(s)\,ds + \tilde{f}(t),$$

where $\tilde{y}(t) = y(t)e^{-\theta_\lambda t}$, and

$$\tilde{a}(t,s) := \frac{p(s)-\lambda}{1-p(t)} r_\lambda(t-s)e^{-\theta_\lambda(t-s)}, \quad \tilde{f}(t) := \frac{1-\lambda}{1-p(t)} z(t)e^{-\theta_\lambda t}. \tag{40}$$

We proceed by demonstrating each of the hypotheses of Theorem 3. A consequence of (39) is that there is $M > 0$ such that $|r_\lambda(u)e^{-\theta_\lambda u}| \le M$ for all $u \ge 0$. Firstly for every $S > 0$,

$$\begin{aligned}\limsup_{t\to\infty}\int_0^S |\tilde{a}(t,t-u)|\,du &= \limsup_{t\to\infty}\frac{1}{1-p(t)}\int_0^S r_\lambda(u)e^{-\theta_\lambda u}|p(t-u)-\lambda|\,du\\ &\le \limsup_{t\to\infty}\frac{M}{1-\lambda}\int_0^S |p(t-u)-\lambda|\,du\\ &\le \limsup_{t\to\infty}\frac{M}{1-\lambda}\int_{t-S}^t |p(\sigma)-\lambda|\,d\sigma = 0,\end{aligned}$$

because of (36). Hence (6) and (7) hold with $b = c = 0$. Let $S > 0$. Then

$$\begin{aligned}\limsup_{t\to\infty}\int_S^{t-S} |\tilde{a}(t,s)|\,ds &= \limsup_{t\to\infty}\int_S^{t-S}\frac{|p(s)-\lambda|}{1-p(t)} r_\lambda(t-s)e^{-\theta_\lambda(t-s)}\,ds\\ &= \limsup_{t\to\infty}\int_S^{t-S}\frac{|p(t-u)-\lambda|}{1-p(t)} r_\lambda(u)e^{-\theta_\lambda u}\,du\\ &\le \frac{M}{1-\lambda}\limsup_{t\to\infty}\int_S^{t-S}|p(t-u)-\lambda|\,du\\ &= \frac{M}{1-\lambda}\limsup_{t\to\infty}\int_S^{t-S}|p(s)-\lambda|\,ds\\ &\le \frac{M}{1-\lambda}\int_S^\infty |p(s)-\lambda|\,ds.\end{aligned}$$

Because of (36), the right-hand side of this equality tends to zero as $S \to \infty$, establishing (8). Next, let $T > 0$. Then

$$\tilde{a}(t,s) = \frac{p(s)-\lambda}{1-p(t)} r_\lambda(t-s)e^{-\theta_\lambda(t-s)} \to \frac{p(s)-\lambda}{1-\lambda} L_\lambda$$

as $t \to \infty$, uniformly for $0 \le s \le T$. Hence (9) is satisfied for $j : [0, \infty) \to \mathbb{R}$ given by

$$j(s) = \frac{L_\lambda}{1-\lambda}(p(s) - \lambda), \quad s \ge 0.$$

The hypothesis (36) ensures that j is integrable. Lastly (34) and (38) imply that

$$z(t)e^{-\theta_\lambda t} \to \frac{L_\lambda}{1-\lambda} \int_0^\infty [1 - p(s)] f(s) e^{-\theta_\lambda s}\, ds \quad \text{as } t \to \infty.$$

All the hypotheses of Theorem 3 have been shown to hold, and we conclude that $\lim_{t\to\infty} \tilde{y}(t)$ exists and satisfies

$$\lim_{t\to\infty} \tilde{y}(t) = \frac{L_\lambda}{1-\lambda} \int_0^\infty \Big\{ (1 - p(s)) f(s) e^{-\theta_\lambda s} + (p(s) - \lambda)\tilde{y}(s) \Big\}\, ds. \tag{41}$$

$\boxed{QED}$

13 Remark. If $f(t) > 0$ for all $t \ge 0$, then $y(t) > 0$ for all $t \ge 0$, and therefore $\lim_{t\to\infty} y(t)e^{-\theta_\lambda t} \ge 0$. It follows from (41) that $\lim_{t\to\infty} y(t)e^{-\theta_\lambda t} > 0$ if $p(t) \ge \lambda$ for all $t \ge 0$: hence in this case $y(t)$ grows or decays at exactly the same rate as $e^{\theta_\lambda t}$.

Next we describe the rate of decay of solutions in the case that (29) is true. Additional assumptions on k are required.

14 Theorem. *Suppose that* (25) *and* (27) *hold. Assume that* (29) *holds, and that* k *is in* $\mathcal{U}(\mu)$. *If* $\lim_{t\to\infty} f(t)/k(t)$ *exists, then* $\lim_{t\to\infty} y(t)/k(t)$ *also exists and*

$$\lim_{t\to\infty} \frac{y(t)}{k(t)} = \frac{1}{1-c_\lambda} \left[\lim_{t\to\infty} \frac{f(t)}{k(t)} + \frac{1}{1-\lambda} \int_0^\infty y(s)k(s)e^{-\mu s}\, ds \right], \tag{42}$$

where

$$c_\lambda = \frac{1}{1-\lambda} \int_0^\infty k(t)e^{-\mu t}\, dt < 1. \tag{43}$$

Proof. We start by dividing (1) by $k(t)$ to get the nonconvolution equation

$$\hat{y}(t) = \int_0^t \hat{a}(t,s)\hat{y}(s)\, ds + \hat{f}(t),$$

where $\hat{y}(t) = y(t)/k(t)$, $\hat{f}(t) = f(t)/k(t)$ and

$$\hat{a}(t,s) := \frac{1}{1-p(t)} \frac{k(t-s)k(s)}{k(t)}. \tag{44}$$

We demonstrate each of the hypothesis of Theorem 3, as in the proof of Theorem 12. Firstly, using (18), (20) and (25), we deduce that

$$\begin{aligned}\limsup_{t\to\infty}\int_0^S |\hat{a}(t,t-u)|\,du &= \lim_{t\to\infty}\frac{1}{1-p(t)}\limsup_{t\to\infty}\int_0^S \frac{k(t-u)k(u)}{k(t)}\,du\\ &= \frac{1}{1-\lambda}\int_0^S k(u)\lim_{t\to\infty}\frac{k(t-u)}{k(t)}\,du\\ &= \frac{1}{1-\lambda}\int_0^S k(u)e^{-\mu u}\,du\\ &\to \frac{1}{1-\lambda}\int_0^\infty k(u)e^{-\mu u}\,ds \quad \text{as } S\to\infty.\end{aligned}$$

It follows from this and (29) that the condition (6) holds. Similarly (7) is true with the constant c_λ given in (43). Due to (29), $0 \le c_\lambda < 1$. By Proposition 9,

$$\begin{aligned}\limsup_{t\to\infty}\int_S^{t-S} |\hat{a}(t,s)|\,ds &= \limsup_{t\to\infty}\int_S^{t-S}\frac{1}{1-p(t)}\frac{k(t-s)k(s)}{k(t)}\,ds\\ &= \lim_{t\to\infty}\frac{1}{1-p(t)}\limsup_{t\to\infty}\int_S^{t-S}\frac{k(t-s)k(s)}{k(t)}\,ds\\ &\to 0 \quad \text{as } S\to\infty,\end{aligned}$$

and therefore (8) is established. Let $T > 0$. We observe from (20) that

$$\hat{a}(t,s) = \frac{k(s)}{1-p(t)}\frac{k(t-s)}{k(t)} \to \frac{k(s)}{1-\lambda}e^{-\mu s} \quad \text{as } t\to\infty, \tag{45}$$

uniformly for $0 \le s \le T$. Then (9) is satisfied with

$$j(s) = \frac{k(s)e^{-\mu s}}{1-\lambda},$$

which by (18) is integrable on $[0,\infty)$.

We conclude from Theorem 3 that $\lim_{t\to\infty}\hat{y}(t)$ exists and is given by (42). $\boxed{QED}$

15 Remark. If $f(t) > 0$ for all $t \ge 0$, (42) implies that $\lim_{t\to\infty} y(t)/k(t) > 0$ and hence $y(t)$ that decays to zero at the same rate as $k(t)$.

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